# Introduction to Supersymmetry

## 1: Formalism of SUSY

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Among models of electroweak symmetry breaking and physics beyond the Standard Model

Supersymmetry (SUSY)

provides one of the most important organizing principles.

In this set of lectures, I will explain the formal basis of SUSY and its application to models of elementary particle physics.

A very useful reference on this subject is:

S. Martin, "A Supersymmetry Primer", hep-ph/9709356

Textbooks on SUSY in elementary particle physics by Baer and Tata and by Dreiner, Haber, and Martin will appear soon.

The outline of these lectures will be:

1. Formalism of SUSY

   symmetry relations of SUSY, construction of Lagrangians

2. The Minimal Supersymmetric Standard Model

   particle content of the MSSM, the Lagrangian, description of symmetry-breaking

3. SUSY spectrum and reactions

   SUSY mass spectrum, illlustrative SUSY reactions, theory of SUSY-breaking parameters

4. Higgs and dark matter in SUSY

   EWSB in the MSSM, Higgs spectrum, dark matter candidates and theory of the dark matter mass density

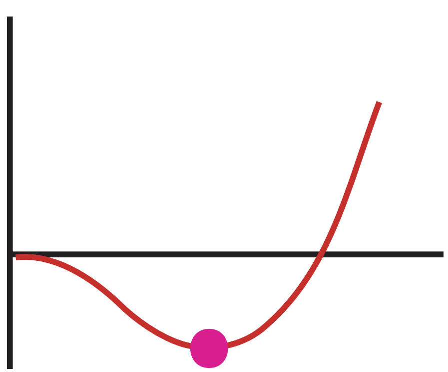
We might begin our study by addressing the most important problem with the minimal form of the Standard Model (MSM):

In the MSM, all masses arise from the spontaneous breaking of SU(2)xU(1). This in turn is due to the vacuum expectation value ("vev") of the Higgs scalar field $\varphi$.

To stabilize a nonzero vev, we postulate a potential

$$V = \mu^2 |\varphi|^2 + \lambda |\varphi|^4$$

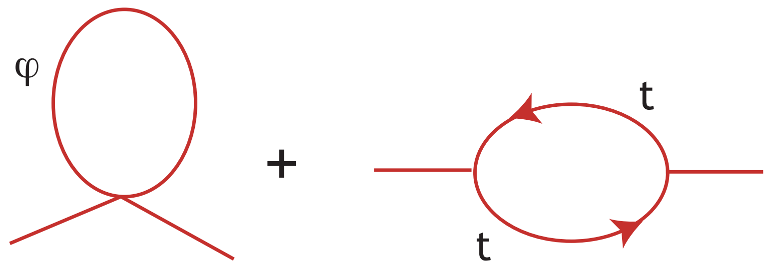
with $\mu^2 < 0$ .



Why is $\mu^2 < 0$ ? "Because it is."

Either sign of $\mu^2$ is possible in principle; there is no preference.

$\mu^2$ receives large additive radiative corrections from




$$\mu^2 = \mu^2_{\rm bare} + \frac{\lambda}{8\pi^2}\Lambda^2 - \frac{3y_t^2}{8\pi^2}\Lambda^2 + \cdots$$

where $\Lambda$ is the largest momentum at which the MSM is still valid. This already says that the criterion $\mu^2 < 0$ is not a simple condition on the bare value of $\mu^2$.

If $\Lambda \sim m_{\rm Pl} \sim 10^{19}$ GeV , $|\mu| \sim 100$ GeV ,
this formalism requires cancellations in the first 36 decimal places. This is called the "gauge hierarchy problem".

There are two ways to solve this problem:

- Lower the cutoff $\Lambda$ to about 1 TeV

For example, postulate that $\varphi$ is not an elementary field but rather is a bound state. Then we need new interactions at 1 TeV ("technicolor") to form this bound state.

Today, this approach is disfavored. Typically, technicolor models give large electroweak corrections and large flavor-changing neutral current amplitudes.

- Insist that $\varphi$ is a fundamental scalar field, but postulate a symmetry that constrains its potential

In particular, this symmetry should forbid the mass term $\mu^2|\varphi|^2$
Violation of the symmetry will re-introduce this term, but hopefully in such a way that we can compute the sign and magnitude.

$\delta\varphi = \epsilon v$ shift symmetry ➤ little Higgs models

$\delta\varphi = \epsilon \cdot A$ gauge symmetry ➤ extra dimension models

$\delta\varphi = \epsilon \cdot \psi$ chiral symmetry ➤ supersymmetry models

From here on, we choose the last option and follows its implications to the end.

So, in the rest of this lecture, I would like to work out the symmetry structure that includes the transformation

$$\delta_\epsilon \phi = \epsilon \cdot \psi$$

where $\phi$ is a complex scalar field and $\psi$ is a spin-$\frac{1}{2}$ field. $\epsilon_\alpha$ is a spin-$\frac{1}{2}$ parameter, represented classically by an anticommuting number. In quantum theory, this transformation is generated by

$$[\epsilon \cdot Q, \phi] = \epsilon \cdot \psi$$

where $Q_\alpha$ is a conserved charge: $[Q_\alpha, H] = 0$

This structure looks innocent, but it is not.

Consider the quantity: $\{Q_\alpha, Q_\beta^\dagger\}$

This object has the following properties:

it is a 4-vector: $\{Q_\alpha, Q_\beta^\dagger\} = 2\gamma^m_{\alpha\beta} R_m$ $\quad *$

it is conserved: $[\{Q_\alpha, Q_\beta^\dagger\}, H] = 0$

it is nonzero:

$$\begin{aligned} \langle\psi| \{Q_\alpha, Q_\alpha^\dagger\} |\psi\rangle &= \langle\psi| Q_\alpha Q_\alpha^\dagger |\psi\rangle + \langle\psi| Q_\alpha^\dagger Q_\alpha |\psi\rangle \\ &= \| Q_\alpha^\dagger |\psi\rangle \|^2 + \| Q_\alpha |\psi\rangle \|^2 \end{aligned}$$

which vanishes only if $Q_\alpha$ and $Q_\alpha^\dagger$ annihilate all states.

So, $R_m$ is a nontrivial conserved 4-vector charge.

* why not a scalar ? see below ...

In a relativistic theory , conservation of energy and momentum already imposes severe restrictions on scattering,

e.g., 2 ➤ 2 scattering amplitudes depend only on 1 parameter, the CM polar angle.

Coleman and Mandula proved that, if there is a second conserved vector, the S-matrix must be trivial:  S = 1.

So, we have no choice: We must set $R^m = P^m$

Then, e.g., $\{Q_1, Q_1^\dagger\} = 2(E - P^3)$

This equation has important implications:

We cannot supersymmetrize just the Higgs field, leaving most of the MSM unchanged. To build a theory with SUSY, all fields in the theory must transform under SUSY.

Even the minimal SUSY extension of the MSM must double the number of particles and fields. This leads to a quite complex theory, but also one with an interesting structure with many implications for the theory of Nature.

To go further, we need to understand 4-d relativistic fermions a little better.

There are two basic spin-1/2 representations of the Lorentz group. Each is 2-dimensional

$$\psi_L \rightarrow (1 - i\vec{\alpha}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2)\psi_L$$

$$\psi_R \rightarrow (1 - i\vec{\alpha}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2)\psi_R$$

Let $c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ; then

$\psi_{1L}^T c \psi_{2L} = -\epsilon_{\alpha\beta}\psi_{1L\alpha}\psi_{2L\beta}$ is Lorentz invariant

$-c\psi_L^*$ transforms like $\psi_R$

C or P carries $\psi_L \leftrightarrow \psi_R$ , but these are not symmetries of the Standard Model.

A Dirac fermion can be written as a pair of L-fermions:

$$\Psi = \begin{pmatrix} \psi_{1L} \\ \psi_{2R} \end{pmatrix} = \begin{pmatrix} \psi_{1L} \\ -c\psi_{2L}^* \end{pmatrix}$$

If we represent $\quad \gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \overline{\sigma}^m & 0 \end{pmatrix} \qquad \begin{aligned} \sigma^m &= (1, \vec{\sigma})^m \\ \overline{\sigma}^m &= (1, -\vec{\sigma})^m \end{aligned}$

The Dirac Lagrangian becomes $\qquad c\sigma^m = (\overline{\sigma}^m)^T c$

$$\begin{aligned} \mathcal{L} &= \overline{\Psi} i\gamma \cdot \partial \Psi - M\overline{\Psi}\Psi \\ &= \psi_{1L}^\dagger i\overline{\sigma} \cdot \partial \psi_{1L} + \psi_{2L}^\dagger i\overline{\sigma} \cdot \partial \psi_{2L} \\ &\qquad -(m\psi_{1L}^T c\psi_{2L} - m^*\psi_{1L}^\dagger c\psi_{2L}^*) \end{aligned}$$

with m = M. In general, m can be a complex number.

Note that $\qquad \psi_{1L}^T c\psi_{2L} = \psi_{2L}^T c\psi_{1L}$

$$(\psi_{1L}^T c\psi_{2L})^* = \psi_{2L}^\dagger (-c)\psi_{1L}^* = -\psi_{1L}^\dagger c\psi_{2L}^*$$

Actually, the most general Lagrangian for massive 4-d fermions has the form

$$\mathcal{L} = \psi_k^\dagger i\overline{\sigma} \cdot \partial \psi_k - (m_{jk}\psi_j^T c \psi_k - m_{jk}^* \psi_j^\dagger c \psi_k^*)$$

(Here and henceforth, I drop the subscript L.)

$$m_{jk} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$ gives a Dirac fermion

with a conserved fermion number

$$Q\psi_1 = +\psi_1 \quad Q\psi_2 = -\psi_2$$

$m_{jk} = m\delta_{jk}$ gives Majorana fermions

In general, m is a complex symmetric matrix, a mixture of Majorana and Dirac mass terms.

The SUSY charges are 4-d fermions. We can write the $Q_\alpha$ as L-fermions, the $Q_\alpha^\dagger$ as R-fermions. The minimal SUSY has two of each: $\alpha = 1, 2$

Now it is clear that $\{Q_\alpha, Q_\beta^\dagger\}$ has no scalar component. The mimimal SUSY algebra in 4-d is then:

$$\{Q_\alpha, Q_\beta^\dagger\} = 2\sigma^m_{\alpha\beta} P_m$$

As an action on fields, with anticommuting L-fermion parameters $\xi, \eta$ , this takes the form:

$$[\delta_\xi, \delta_\eta] = 2i[\xi^\dagger \overline{\sigma}^m \eta - \eta^\dagger \overline{\sigma}^m \xi]\,\partial_m$$

Now we can look for representations of this algebra.

The simplest representation includes a complex scalar field $\phi$ and an L-fermion field $\psi$ . This is the chiral supermultiplet. Its particle content includes

2 bosons $\phi, \phi^*$ + 2 fermions $\psi_L, (\psi^*)_R$

For convenience in writing the algebra, I add a second complex field $F$ that will have no associated particles.

$$\delta_\xi \phi = \sqrt{2}\xi^T c\psi$$

$$\delta_\xi \psi = \sqrt{2} i\sigma^n c\xi^* \partial_n \phi + \sqrt{2}\xi F$$

$$\delta_\xi F = -\sqrt{2} i\xi^\dagger \overline{\sigma}^m \partial_m \psi$$

We need to check that this transformation

1. satisfies the fundamental commutation relations
2. leaves a suitable Lagrangian invariant

Check #1 on $\phi$ :

$$
\begin{aligned}
[\delta_\xi, \delta_\eta]\phi = & - \delta_\xi(\sqrt{2}\eta^T c\psi) - (\xi \leftrightarrow \eta) \\
= & \ \ \sqrt{2}\eta^T \ \sqrt{2} i\sigma^n c\xi^* \partial_n \phi - (\xi \leftrightarrow \eta) \\
= & \ \ 2i\eta^T c\sigma^n c\xi^* \partial_n \phi - (\xi \leftrightarrow \eta) \\
= & \ \ 2i[\xi^\dagger \overline{\sigma}^n \eta - \eta^\dagger \overline{\sigma}^n \xi] \ \partial_n \phi
\end{aligned}
$$

The check on $F$ is equally easy. The check on $\psi$ requires an extra trick, but it works.

Check #2 on $\mathcal{L} = \partial^m \phi^* \partial_m \phi + \psi^\dagger i \overline{\sigma} \cdot \partial \psi + F^* F$

using integration by parts freely under $\int d^4x$

$$
\begin{aligned}
\delta_\xi \mathcal{L} = \;& \partial^m \phi^* \partial_m (\sqrt{2} \xi^T c \psi) + (-\sqrt{2} \partial^m \psi^\dagger c \xi^*) \partial_m \phi \\
& + \psi^\dagger i \overline{\sigma}^m \partial_m \left[ \sqrt{2} i \sigma^n c \xi^* \partial_m \phi + \sqrt{2} \xi F \right] \\
& + [\sqrt{2} i \partial_n \phi^* \xi^T c \sigma^n + \sqrt{2} \xi^\dagger F^*] i \overline{\sigma}^m \psi \\
& + F^* [-\sqrt{2} i \xi^\dagger \overline{\sigma}^m \partial_m \psi] + [\sqrt{2} i \partial_m \psi^\dagger \overline{\sigma}^m \xi] F \\
= \;& -\phi^* \sqrt{2} \xi^T c \partial^2 \psi + \sqrt{2} \partial_n \phi^* \xi^T c \overline{\sigma}^n \sigma^m \partial_n \partial_m \psi \\
& + \sqrt{2} \psi^\dagger c \xi^* \partial^2 \phi - \sqrt{2} \psi^\dagger \overline{\sigma}^m \sigma^n c \xi^* \partial_m \partial_n \phi \\
& + \sqrt{2} i \psi^\dagger \overline{\sigma}^m \partial_m F \xi + \sqrt{2} i \partial_m \psi^\dagger \overline{\sigma}^m \xi F \\
& + \sqrt{2} i \xi^\dagger F^* \overline{\sigma}^m \partial_m \psi - \sqrt{2} i F^* \xi^\dagger \overline{\sigma}^m \partial_m \psi \\
= \;& 0
\end{aligned}
$$

So far, our theory is trivial. But we can give it interactions in a straightforward way.

Let $W(\phi)$ be an analytic function of $\phi$, the "superpotential".

Let $$\mathcal{L}_W = F\frac{\partial W}{\partial \phi} - \frac{1}{2}\psi^T c\psi\frac{\partial^2 W}{\partial \phi^2}$$

$$\begin{aligned}\delta_\xi \mathcal{L}_W = & F\frac{\partial^2 W}{\partial \phi^2}(\sqrt{2}\xi^T c\psi) - \sqrt{2}F\xi^T c\psi\frac{\partial^2 W}{\partial \phi^2} \\ & -\sqrt{2}i\xi^\dagger\overline{\sigma}^m\partial_m\psi\frac{\partial W}{\partial \phi} - \psi^T c\sqrt{2}i\sigma^n c\xi^*\partial_n\phi\frac{\partial^2 W}{\partial \phi^2} \\ & -\psi^T c\psi\frac{\partial^3 W}{\partial \phi^3}\sqrt{2}\xi^T c\psi\end{aligned}$$

the 2nd line rearranges to: $-\sqrt{2}i\xi^\dagger\overline{\sigma}^n(\partial_n\psi\frac{\partial W}{\partial \phi} + \psi\partial_n\phi\frac{\partial^2 W}{\partial \phi^2})$

the 3rd line is proportional to $\psi_\alpha\psi_\beta\psi_\gamma$, which vanishes.

In fact, the most general (renormalizable) supersymmetric Lagrangian involving only spin 0 and spin 1/2 fields is

$$\mathcal{L} = \partial^m \phi_k^* \partial_m \phi_k + \psi_k^\dagger i\overline{\sigma}^m \partial_m \psi_k + F_k^* F_k + (\mathcal{L}_W + h.c.)$$

where $\mathcal{L}_W$ is built from an analytic function $W(\phi_k)$

$$\mathcal{L}_W = F_k \frac{\partial W}{\partial \phi_k} - \frac{1}{2} \psi_j^T c \psi_k \frac{\partial W}{\partial \phi_j \partial \phi_k}$$

The $F_k$ are Lagrangian multipliers with constraint equations

$$F_k^* = -\frac{\partial W}{\partial \phi_k}$$

Eliminating the $F_k$ using these equations, we find the potential

$$V_F = \sum_k \left| \frac{\partial W}{\partial \phi_k} \right|^2$$

It is important that V ≥ 0 and V = 0 only if all $F_k$= 0 .

Recall that $\langle 0| \{Q_\alpha, Q_\alpha^\dagger\} |0\rangle = \langle 0| (H - P^3) |0\rangle$

This is ≥ 0 and is = 0 only if $Q_\alpha |0\rangle = Q_\alpha^\dagger |0\rangle = 0$

Now consider

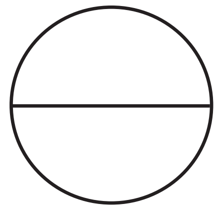
$$\begin{aligned}\langle 0| [\xi^T c Q_\alpha, \psi_k] |0\rangle &= \langle 0| \sqrt{2} i \sigma^n \xi^* \partial_n \phi_k + \xi F_k |0\rangle \\ &= \xi \langle 0| F_k |0\rangle\end{aligned}$$

If the vacuum is supersymmetric, this must vanish.
Unbroken SUSY then implies

< H > = 0 <F> = 0

Becasue H ≥ 0, SUSY can be spontaneously broken only if it is impossible to find a state where <H> = 0.

These are exact results, and so it must follow that the vacuum energy vanishes in perturbation theory,

 = 0

I would like to show you another type of cancellation that is also seen in SUSY perturbation theory. Consider

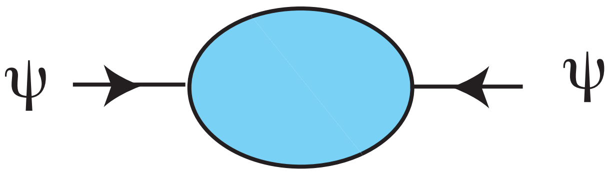
$$W = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3$$

Eliminating F, we find

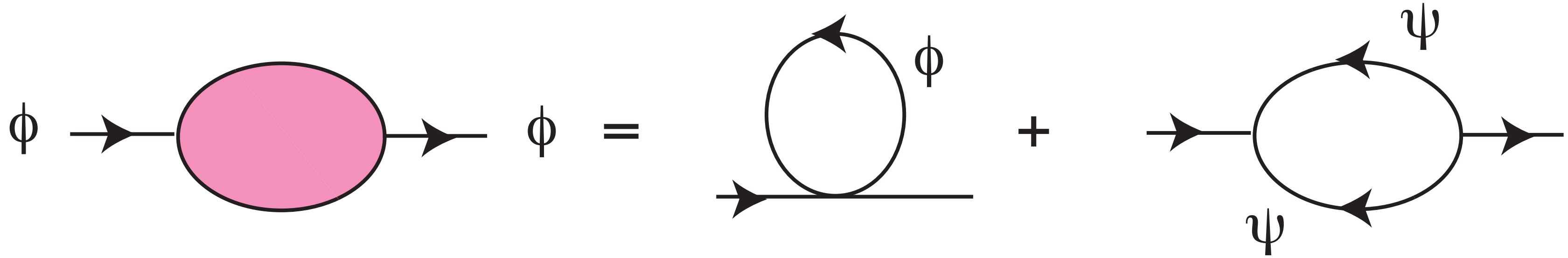
$$\mathcal{L} = \partial^m \phi^* \partial_m \phi + \psi^\dagger i \overline{\sigma}^m \partial_m \psi - |m\phi + \lambda\phi^2|^2$$
$$-\frac{1}{2}(m + 2\lambda\phi)\psi^T c\psi + \frac{1}{2}(m + 2\lambda\phi)^* \psi^\dagger c\psi^*$$

$\phi$ and $\psi$ have equal masses = |m|.

From our previous experience, we might expect an additive radiative correction to the mass. Check this at m = 0 :




= 0 (no diagrams)

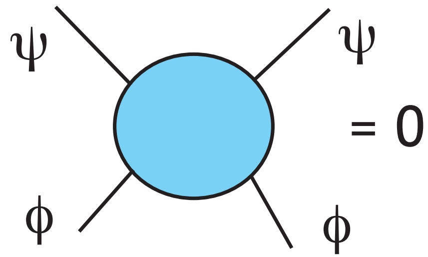


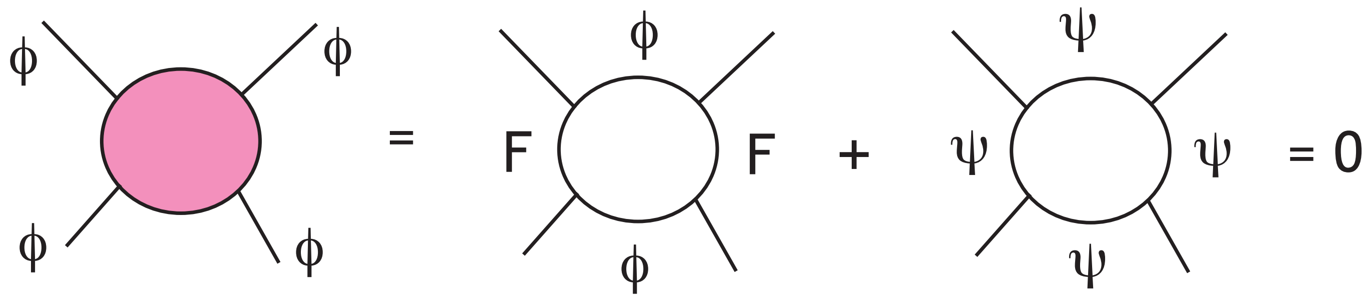

$$= -4i\lambda^2 \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2}$$

$$+\frac{1}{2}(-2i\lambda)(+2i\lambda) \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}[\frac{i\sigma \cdot p}{p^2} c \frac{i\sigma^T \cdot (-p)^T}{p^2} c]$$

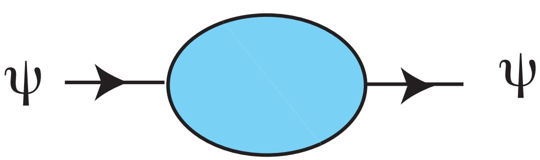
$$= 0$$

In fact, it may be shown quite generally that the superpotential W receives no radiative corrections. In 1 loop:

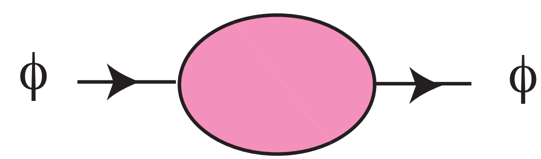



The field strength renormalization can be nonzero




$$= i\delta Z\overline{\sigma}\cdot p$$




$$= -i\delta Z\, p^2$$

so the fields in W can be rescaled by radiative corrections.

This cancellation is more obvious in a manifestly supersymmetric perturbation theory (supergraphs).

In a supersymmetric generalization of the MSM, the non-renormalization of the superpotential would eliminate the dangerous additive radiative corrections to the Higgs mass.

This makes SUSY a good starting point for the construction of models of physics beyond the SM.  We will turn to that model construction in the next lecture.